Solution to Assignment 1

1. Consider the function $\varphi(x) = x^{-a}$ where a is positive for $x \in (0,1]$ and set $\varphi(0) = 1$ so that φ is a well-defined function on $[0,1]$. Show that φ is not integrable on $[0,1]$. This is the simplest example of an unbounded function.

   Solution. Assume on the contrary that φ is integrable on $[0,1]$. Given any number $\varepsilon > 0$, there is a partition P such that
$$|\sum_j \varphi(x_j^*)\Delta x_j - I| < \varepsilon \ ,$$
   for any tags on P. (We don't care about the length of P.) Equivalently,
$$-\varepsilon \le \sum_j \varphi(x_j^*)\Delta x_j - I \le \varepsilon \ .$$
   Taking $\varepsilon = 1$, say, we have
$$\sum_j \varphi(x_j^*)\Delta x_j \le 1 + I \ .$$
   We dispose all summands in the summation above except the first summand to get
$$\frac{1}{(x_1^*)^a}\Delta x_1 = \varphi(x_1^*)\Delta x_1 \le 1 + I \ .$$
   The right hand of this inequality is a finite number. However, if we choose the tag x_1^* very close to 0, the left hand side could be arbitrarily large, hence this inequality cannot be true. The contradiction shows that φ is not integrable.

   Note. Nonetheless, for $a \in (0,1)$ φ is improperly integrable.

2. Consider the function H in $\mathbb{R}^2$ defined by $H(x,y) = 1$ whenever x, y are rational numbers and equals to 0 otherwise. Show that H is not integrable in any rectangle.

   Solution. Let P be any partition of the rectangle. By choosing tags points (x^*, y^*) where x^* and y^* are rational numbers,
$$\sum_{j,k} H(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} \Delta x_j \Delta y_k$$
   which is equal to the area of R. On the other hand, by choosing the tags so that x^* is irrational, $H(x^*, y^*) = 0$ so that
$$\sum_{j,k} H(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} 0 \times \Delta x_j \Delta y_k = 0 \ .$$
   Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit. We conclude that H is not integrable.

3. Let $f = f(x,y)$ be a bounded function defined in $R = [0,1] \times [0,1]$ which is 0 everywhere except at a point $(0,0)$. Show that f is integrable in R with integral equal to 0.

   Solution. Let M satisfy $|f(x,y)| \le M$ for all (x,y). Let P be any partition of R. The Riemann sum of this partition is equal to
$$\sum_{j,k} f(x_j^*, y_k^*)\Delta x_j \Delta y_k = f(x_1^*, y_1^*)\Delta x_1 \Delta y_1 \ .$$

Therefore,

$$\begin{aligned} \left|\sum_{j,k} f(x_j^*, y_k^*)\Delta x_j \Delta y_k - 0\right| &= |f(x_1^*, y_1^*)\Delta x_1 \Delta y_1| \\ &\leq M\|P\|^2, \end{aligned}$$

which shows that the Riemann sums tend to 0 as $\|P\|$'s tend to 0. We conclude that f is integrable and

$$\iint_R f(x,y)\,dA = 0\ .$$

4. Let $g = g(x,y)$ be a bounded function defined in $R = [0,1]\times[0,1]$ which is 0 everywhere except along the line $x = 1/2$. Show that f is integrable in R with integral equal to 0.

Solution. The proof is similar to the previous one. Let P be any partition of the rectangle. Assume first that $1/2 \in (x_{j_0-1}, x_{j_0})$. (That is, $1/2$ lies in the interior of some subinterval.) We let $\mathcal{A}$ denote the collection of subrectangles of P of the form $[x_{j_0-1}, x_{j_0}]\times[y_{k-1}, y_k]$. Then

$$\begin{aligned} \left|\sum_{j,k} f(x_j^*, y_k^*)\Delta x_j \Delta y_k - 0\right| &= \left|\sum_{\mathcal{A}} f(x_{j_0}^*, y_k^*)\Delta x_{j_0} \Delta y_k\right| \\ &= \left|\sum_{k} f(x_{j_0}^*, y_k^*)\Delta x_{j_0} \Delta y_k\right| \\ &\leq M\Delta x_{j_0}\sum_k \Delta y_k \\ &\leq M\|P\|, \end{aligned}$$

which shows that the Riemann sums tend to 0 as $\|P\|$'s tend to 0. The proof is similar when $1/2 = x_{j_0}$ for some j_0. (That is, $1/2$ is the endpoint of two consecutive subintervals.)